JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 31, 147–153 (1970)

The Convergence of Padé Approximants of Meromorphic Functions*

J. NUTTALL

Physics Department, Texas A & M University, College Station, Texas 77843

Submitted by S. Ulam

It is proved that the $[N, N + J]$ Padé approximants to any meromorphic function converge in measure within any bounded region of the complex plane as N approaches infinity.

1. INTRODUCTION

Padé approximants have been applied in an effort to overcome convergence problems in many areas, sometimes with considerable success [1]. However, except in the case of series of Stieltjes, very little has been proved about their convergence. The theorems that exist are unsatisfactory in that they postulate certain properties of the Padé approximants, and it is not known whether these are satisfied [2].

We have previously pointed out the connection between Padé approximants and approximations derived from stationary variational principles [3]. We have shown that, in some cases, variational principles lead to convergence in measure [4] rather than convergence in the normal sense [5], and have suggested that it may be possible to prove that Padé approximants converge in measure. For a limited class of entire functions this has already been done [6], and in this paper the result is extended to any meromorphic function. A precise statement of the theorem proved here is

THEOREM. *Let $P_N(z)$ be the $[N, N + J]$ Padé approximant to a meromorphic function $F(z)$, and $\mathscr{D}$ be a closed, bounded region of the complex plane. Then, given any ϵ, $\delta > 0$, there exists N_0 such that, for all $N > N_0$*

$$| P_N(z) - F(z)| < \epsilon$$

for all $z \in \bar{\mathscr{D}}_N$, where *$\bar{\mathscr{D}}_N \subset \mathscr{D}$ and the measure of $\mathscr{D} - \bar{\mathscr{D}}_N$ is less than δ.*

* Supported in part by the U.S. Air Force Office of Scientific Research, Office of Aerospace Research, under grant No. 69-1817.

In the proof we shall assume without loss of generality that the region $\mathscr{D}$ is given by $|z| \leqslant 1$, and that $J \geqslant 0$. The proof depends on two important lemmas, one an algebraic property of Padé approximants and the other related to a classical maximum problem. We shall first state the lemmas and prove the theorem, leaving the proof of the lemmas to follow.

Before stating the first lemma, we need to introduce some notation. If $Q(z)$ is a polynomial of the form

$$Q(z) = \sum_{j=0}^{N} q_j z^j \tag{1}$$

we mean by $[Q]_a^b$ the expression

$$[Q]_a^b = \sum_{j=a}^{b} q_j z^j. \tag{2}$$

If an index is omitted it is taken to be 0, N for a, b, respectively.

LEMMA 1. *Let a function $f(z)$ have the form*

$$f = g + \frac{Q}{R} \tag{3}$$

where

$$g(z) = \sum_{j=0}^{2N+J} g_j z^j \tag{4}$$

and Q, R are polynomials of order $m - 1$, m, respectively. Write the $[N, N+J]$ Padé approximant to f as B/D, where B, D are polynomials of order $N+J$, N, respectively. Then, if $m < N+1$,

$$\frac{B}{D} - f = -\frac{[Dg]_{2N+J+1}}{D} + \frac{[R[Dg]^{2N+J}]_{2N+J+1}}{(DR)} \tag{5}$$

The second lemma relates to a polynomial $D(z) \equiv \prod_{i=1}^{N} (z - z_i)$ and states

LEMMA 2. *Whatever the values of z_i, N, the inequality*

$$|D(z)| \leqslant x^N$$

holds in a region of the complex z-plane whose measure is never greater than πx^2.

2. PROOF OF THEOREM

Any meromorphic function $F(z)$ may be written in the form [7]

$$F(z) = e(z) + \sum_{j=0}^{\infty} [G_j(z) + K_j(z)] \tag{6}$$

where $e(z) = \sum_{j=0}^{\infty} e_j z^j$ is entire. The j-th term in the sum (6) is related to the j-th pole ζ_j of $F(z)$, with $|\zeta_j| \geqslant |\zeta_{j-1}|$, and

$$G_j(z) = \frac{b_{jk}}{(z - \zeta_j)^k} + \cdots + \frac{b_{j1}}{z - \zeta_1}. \tag{7}$$

The function $K_j(z)$ is a polynomial which may be chosen in such a way that, given any $\rho > 0$ we may find $n(\rho)$, $\epsilon_j > 0$, with the property

$$|G_j(z) + K_j(z)| < \epsilon_j \qquad \text{if} \qquad |z| \leqslant \rho, \quad j \geqslant n(\rho) \tag{8}$$

The sum $\sum_{j=0}^{\infty} \epsilon_j$ is convergent with value C.

Suppose we are given the δ mentioned in the statement of the theorem. We choose x so that $\pi x^2 < \frac{1}{2}\delta$. Now set $\rho = 8/x$ and define polynomials Q, R, integer m, by

$$\frac{Q}{R} = \sum_{j=0}^{n(\rho)} G_j(z) \tag{9}$$

where

$$R = \prod_{i=1}^{m} (z - \alpha_i) \tag{10}$$

The α_i are the ζ_j for $j \leqslant n(\rho)$, repeated if necessary. For small enough δ, ρ will be large and we assume $\rho > 1$.

In the region $|z| \leqslant 1$ the function

$$H(z) \equiv \sum_{j=n(\rho)+1}^{\infty} [G_j(z) + K_j(z)]$$

is analytic and may be expanded as

$$H(z) = \sum_{j=0}^{\infty} h_j z^j. \tag{11}$$

From (8) we deduce that

$$|h_j| \leqslant C\rho^{-j}. \tag{12}$$

Now define the function f of Lemma 1 as

$$f(z) \equiv \sum_{j=0}^{2N+J} (e_j + h_j) z^j + \sum_{j=0}^{n(\rho)} K_j(z) + \frac{Q}{R}. \tag{13}$$

We assume in later developments that N is greater than the sum of m and the highest exponent of z that appears in $K_j(z)$, $j \leqslant n(\rho)$.

The difference between $F(z)$ and $f(z)$,

$$F(z) - f(z) = \sum_{j=2N+J+1}^{\infty} (e_j + h_j)\, z^j, \tag{14}$$

may be made as small as we please uniformly over $|z| \leqslant 1$ by taking N large enough. Consequently, since the $[N, N+J]$ Padé approximants to F and f are identical, to prove the theorem we need only show that the difference between f and its approximant may, by increasing N, be made vanishingly small over all of $|z| \leqslant 1$ except for a set of measure δ. This set depends on N.

Consider the first term of the formula given by Lemma 1 for the difference between $[N, N+J]$ and f, namely $-[Dg]_{2N+J+1}/D$. The numerator is a polynomial in z and contains no more than $2^N N$ terms of the form

$$g_i z^k z_p z_q \cdots z_r$$

with $i \geqslant N+1$, no z_j being repeated in the product in brackets. We divide the z_i into two sets so that

$$|z_i| \leqslant 2, \qquad i \leqslant l$$

$$|z_i| > 2, \qquad i > l.$$

With this information we deduce for $|z| \leqslant 1$ that

$$\begin{aligned} \left| \frac{[Dg]_{2N+J+1}}{D} \right| &\leqslant 2^N N g\, 2^l \left| \prod_{j=1}^{l} (z - z_j) \right|^{-1} \prod_{j=l+1}^{N} \left| \frac{z_j}{z - z_j} \right| \\ &\leqslant 2^{2N} N g \left| \prod_{j=1}^{l} (z - z_j) \right|^{-1} \end{aligned} \tag{15}$$

where g is the maximum value of $|g_i|$, $i = N, \ldots, 2N+J+1$, and from (3) and (13) we have

$$g_i = e_i + h_i, \qquad i \geqslant N. \tag{16}$$

Lemma 2 shows that $|\prod_{j=1}^{l} (z - z_j)|^{-1} > x^{-l}$ except for a set of measure not greater than $\pi x^2 < \frac{1}{2}\delta$. Thus, for all $|z| \leqslant 1$ except a set of measure $< \frac{1}{2}\delta$ we have

$$\left| \frac{[Dg]_{2N+J+1}}{D} \right| \leqslant 2^{2N} N g x^{-l} \leqslant Ng \left(\frac{4}{x} \right)^N. \tag{17}$$

Since $e(z)$ is entire, and h_i satisfies (12) with $\rho = 8/x$, it follows that (17) may be made as small as we please by increasing N far enough. A similar argument leads to the same conclusion for the second term of (5) and the theorem is proved.

3. Proof of Lemma 1

We present a proof of this Lemma kindly provided by Dr. G. A. Baker, Jr. It is shorter and more transparent than the original derivation.

From the definition of the Padé approximant we have

$$\frac{B}{D} - g - \frac{Q}{R} = \mathcal{O}(z^{2N+J+1})$$

so that

$$[BR - QD - g\,DR]^{2N+J} = 0. \tag{18}$$

Now $d \equiv [N, N + J] - f$ may be written as

$$d = \frac{B}{D} - f = \frac{(BR - QD - g\,DR)}{(DR)}$$

$$= \frac{[BR - QD - g\,DR]_{2N+J+1}}{(DR)}.$$

If $m < N + 1$ we have

$$[BR]_{2N+J+1} = [QD]_{2N+J+1} = 0,$$

so that

$$d = -\frac{[g\,DR]_{2N+J+1}}{(DR)}. \tag{19}$$

Now $[g\,DR]_{2N+J+1}$ may be written as

$$[g\,DR]_{2N+J+1} = [R[Dg]_{2N+J+1}]_{2N+J+1} + [R[Dg]^{2N+J}]_{2N+J+1}$$

$$= R[Dg]_{2N+J+1} + [R[Dg]^{2N+J}]_{2N+J+1}$$

which leads to the required result.

4. Proof of Lemma 2

This lemma is closely related to some classical results set forth, for instance, by Pólya and Szegö [8]. It may be stated in terms of the function $\phi(z) = \operatorname{Re} \ln D(z)$, which serves as an electrostatic potential in two dimensions. Let $A(V)$ be the area enclosed by the equipotential $\phi = V$. Then Lemma 2 follows if it can be shown that $A(V)$ is not greater than the value it takes on when $z_1 = z_2 = \cdots = z_N$, say $A_0(V)$. Since $A(V_1) < A(V_2)$ if $V_1 < V_2$, the result follows if we can show that $V_1 \geqslant V_0$ if $A(V_1) = A_0(V_0)$.

A result given in [8] leads to this immediately. It states that the capacity per unit length of two cylinders $\phi = V_1$, $\phi = V_2$ is not less than the capacity of two concentric circular cylinders with the same cross-sectional areas as $\phi = V_1$, $\phi = V_2$, respectively. If the radii of the circular cylinders are r_1, r_2, we have

$$\frac{N}{V_2 - V_1} \geqslant (\ln r_2 - \ln r_1)^{-1}$$

or

$$V_2 - V_1 \leqslant N(\ln r_2 - \ln r_1). \tag{20}$$

We note that $V_0 = N \ln r_1$. Also, since $\phi \sim N \ln |z| + 0(|z|^{-1})$ as $|z| \to \infty$, we deduce that, for large V_2, r_2

$$V_2 \underset{r_2 \to \infty}{\sim} N \ln r_2 + 0(r_2^{-1}).$$

Taking the limit $r_2 \to \infty$ in (20) gives $V_1 \geqslant V_0$ as required.

5. Acknowledgments

The author is grateful to Dr. G. A. Baker, Jr., Dr. J. L. Gammel, Dr. G. D. McCartor, and Dr. R. J. Rubin for helpful discussions.

References

1. G. A. Baker, Jr., "Advances in Theoretical Physics," (K. A. Brueckner, ed.), pp. 1–57, Academic Press, New York, 1965.
2. G. A. Baker, Jr., *contribution to* "The Padé Approximant in Theoretical Physics," (G. A. Baker, Jr. and J. L. Gammel, eds.), Academic Press, New York (to be published).
3. J. Nuttall, *Phys. Rev.* **157** (1967), 1312–1316.
4. P. R. Halmos, "Measure Theory," Van Nostrand, Princeton, N. J., 1950.
5. J. Nuttall, *Ann. Phys.* **52** (1969), 428–443.

6. J. NUTTALL, *contribution to* "The Padé Approximant in Theoretical Physics," (G. A. Baker, Jr. and J. L. Gammel, eds.), Academic Press, New York (to be published).
7. A. I. MARKUSHEVICH, "Theory of Functions of a Complex Variable," Vol. II, p. 300, Prentice-Hall, Englewood Cliffs, N. J., 1965.
8. G. PÓLYA AND G. SZEGÖ, "Isoperimetric Inequalities in Mathematical Physics," Princeton Univ. Press, Princeton, N. J., 1951.